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LETTER TO THE EDITOR

Painlevé analysis of new soliton equations by Hu

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Abstract. We perform the singularity analysis of four new generalized bilinear equations by Hu, of which two possess two-soliton solutions and the other two possess even three-soliton solutions. All four equations fail to pass the Painlevé test for integrability, exhibiting non-dominant movable logarithmic singularities at highest resonances of generic branches.

Recently, Hu [1] proposed a new generalization of Hirota's bilinear equations, derived conditions for the existence of multi-soliton solutions, and quoted six examples of such generalized bilinear equations. Hu [1] pointed out that two of the examples have been known as integrable equations by Tu [2] and by Harada [3], and proved for the other four examples that two equations possessed two-soliton solutions and two equations possessed even three-soliton solutions. Are the four new equations by Hu integrable as well? Existence of a two-soliton solution *does not* guarantee integrability of a bilinear equation, and a three-soliton solution *is only conjectured* to be sufficient for integrability [4]. Moreover, checking conditions for the existence of N-soliton solutions is too tedious even for low N. Therefore, instead of proving the existence of N-soliton solutions for higher and higher N, we will employ another test for integrability, namely, the Painlevé test by Weiss *et al* [5]. In this letter, we will show that, because of bad singularity structure of solutions, none of the four new soliton equations by Hu should be expected to be integrable. We will follow the Weiss–Kruskal algorithm of Painlevé analysis [6], omitting unessential details.

The first nonlinear system to be tested, namely,

$$u_t = u_{xx} + 2uu_x + v_x \qquad v_t = v_{xxx} + 6u_x v_x \tag{1}$$

has a two-soliton solution [1]. This is a normal system of total order five, a hypersurface $\varphi(x,t) = 0$ is non-characteristic for (1) if $\varphi_x \neq 0$, and the general solution of (1) must contain five arbitrary functions of one variable [7]. We analyse singularities of solutions at non-characteristic hypersurfaces only [8], use the Kruskal ansatz [9] $\varphi_x = 1$, and assume the dominant behaviour of solutions of (1) to be algebraic: $u = u_0(t)\varphi^{\alpha} + \dots$ and $v = v_0(t)\varphi^{\beta} + \dots$, where α and β are complex contants, $u_0 v_0 \neq 0$. System (1) admits several branches (i.e. choices of α, β, u_0 and v_0), of which we consider the following two as most striking: (i) $\alpha = \beta = -1$, $u_0 = 1$, $v_0(t)$ is arbitrary, and (ii) $\alpha = -1$, $\beta = -2$, $u_0 = 2$, $v_0 = -2$. In case (i), positions r of resonances are $r = -1, 0, 1, 2, 5$, where $r = -1$ corresponds to arbitrary $\psi(t)$ in $\varphi = x + \psi$, so that (i) is the generic branch. Substituting $u = \sum_{k=0}^{\infty} u_k(t)\varphi^{k-1}$ and $v = \sum_{k=0}^{\infty} v_k(t)\varphi^{k-1}$ into (1), we get recursion relations for u_k and v_k, and then check compatibility conditions at resonances. At $r = 1$ and 2, where arbitrary functions $v_1(t)$ and $u_2(t)$ appear, we get identities. However, a complicated third-order

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differential equation $\psi_{ttt} + \ldots = 0$ for ψ, v_0, v_1 and u_2 arises at $r = 5$, which forces us to modify the Weiss–Kruskal expansions by introducing a logarithmic term at this resonance [6]. Therefore system (1) fails to pass the Painlevé test for integrability. Moreover, branch (ii) has resonances in positions -2, -1, 2 and $\frac{9}{2} \pm \frac{1}{2}\sqrt{33}$, so that we could have stopped the singularity analysis even at the step of finding resonances.

Next we consider the nonlinear system

$$u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + v_x$$
$$v_t = v_{xxx} + 6u_xv_x \tag{2}$$

which has a three-soliton solution [1]. The same way (with the same notations) as for system (1) leads us to the conclusion that (2) fails to pass the Painlevé test as well. The generic branch, where $\alpha = \beta = -1$, $u_0 = 1$, and $v_0(t)$ is arbitrary, has resonances at $r = -1, 0, 1, 1, 3, 5$. At $r = 1$ and 3, where arbitrary functions $u_1(t)$, $v_1(t)$ and $u_3(t)$ appear, compatibility conditions are satisfied identically, but this is not the case at $r = 5$, where a first-order differential equation for ψ, v_0, u_1 and u_3 arises. Moreover, one of several non-generic branches, namely, that with $\alpha = -1$, $\beta = -3$, $u_0 = \frac{10}{3}$ and $v_0 = -\frac{280}{27}$ has resonances in irrational positions, i.e. r is -1 or 3 or any root of $3r^4 - 18r^3 - 107r^2 + 122r + 560 = 0$.

Starting the Painlevé analysis of the system

$$u_{tt} - u_{xx} + u_t^2 - u_x^2 - v = 0$$
$$v_{tt} - v_{xx} + 2v(u_{tt} - u_{xx}) = 0 \tag{3}$$

which has a three-soliton solution [1], we are faced with the absence of any algebraic dominant behaviour of solutions. This is caused by a dominant logarithmic singularity in u. In this respect, system (3) is similar to the sine-Gordon equation [5] and the Liouville equation [10]. The transformation $a(y, z) = \exp u$, $b(y, z) = v$, $y = \frac{1}{2}(t + x)$ and $z = \frac{1}{2}(t - x)$ (cone co-ordinates y and z are used for simplicity) changes (3) into the equivalent system

$$a_{yz} - ab = 0 \qquad a^2b_{yz} - 2ba_ya_z + 2a^2b^2 = 0 \tag{4}$$

which admits several branches of algebraic dominant behaviour of solutions, $a = a_0(z)\varphi^\alpha + \ldots$ and $b = b_0(z)\varphi^\beta + \ldots$, $\varphi = y + \psi(z)$, $\psi_z \neq 0$ since $\varphi = 0$ must be non-characteristic for (4). In the generic branch, where $\alpha = 1$, $\beta = -1$, and $a_0(z)$ and $b_0(z)$ are arbitrary, we find resonances to be $r = -1, 0, 0, 3$ and get at $r = 3$ a complicated third-order differential equation for ψ, a_0 and b_0. Therefore system (4) has a non-dominant logarithmic singularity in its general solution. In non-generic branches, resonances stand in integer positions, but we will not check compatibility conditions, since (4) has failed the Painlevé test in the generic branch.

Lastly we have to test the following generalized bilinear equation

$$(D_t - D_x^5)[(\partial_t - \partial_x^5)\tau] \cdot \tau = 0 \tag{5}$$

where D_t and D_x are Hirota's bilinear operators, ∂_t and ∂_x are partial derivatives, $\tau = \tau(x, t)$. Equation (5) has a two-soliton solution and is suspected of having a three-soliton solution [1]. The transformation $u = \tau_x/\tau$ and $v = (\partial_t\tau - \partial_x^5\tau)/\tau$ changes (5) into the system

$$u_t = \partial_x(\partial_x + u)^4u + v_x$$
$$v_t = v_{xxxxx} + 20u_xv_{xxx} + 10(u_{xxx} + 6u_x^2)v_x \tag{6}$$

which is similar to (1) and (2) in form. The same way (with the same notations) as for (1) shows that (6) fails to pass the Painlevé test as well. The generic branch, where $\alpha = \beta = -1$, $u_0 = 1$, and $v_0(t)$ is arbitrary, has resonances at $r = -1, 0, 1, 1, 2, 2, 3, 3, 5, 9$. Again, at the highest resonance, $r = 9$, we are forced to introduce a logarithmic term. Moreover, (6) possesses many non-generic branches with dominant transcendent singularities, but we will not consider them.

Thus, we have established that, unfortunately, none of the four new soliton equations by Hu [1] pass the Painlevé test for integrability. This is, however, *not* a defect in Hu's generalization of Hirota's bilinear equations but a random feature of some of the examples selected by Hu for [1]. Indeed, two more examples of generalized bilinear equations quoted in [1], namely, the integrable systems by Tu [2] $\{u_t = u_x + 2v,\ v_t = -2uv\}$ and by Harada [3] $\{u_t = u_{xx} + 2uu_x - 2v_x,\ v_t = v_{xx} + 2u_x v\}$, pass the Painlevé test well. We have to conclude that even the existence of a three-soliton solution may *not* lead to integrability. Certainly, correctness of this conclusion depends on the reliability of the Painlevé test. Some integrable equations have no Painlevé property by themselves, but they can be transformed so that the Painlevé property can be restored (e.g. the sine–Gordon equation [5] and the Dym–Kruskal equation [11]). However, general solutions of Hu's new soliton equations possess non-dominant logarithmic singularities which, as is generally believed [6, 12], cannot be removed by any transformations.

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